What is a Crowd for a Mathematician?
Modeling, Simulations, and Analytic Problems

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1. From the Question “What is a Crowd?” to a Modeling Strategy

2. The Kinetic Theory Approach to Crowd Modeling

3. From Microscopic to Macroscopic

1.2 - From “What is a Crowd?” to a Modeling Strategy

Five key questions waiting for an answer

1. *Why a crowd is a “social, hence complex,” system?*

2. *How mathematical sciences can contribute to understand the “behavioral dynamics of crowds”?*

3. *How the crowd behaves in extreme situations such as panic and how models can depict them as well as large deviations (black swans)?*

4. *How multiscale problems can be treated?*

5. *Which are the most challenging research perspectives?*

1.2 - From “What is a Crowd?” to a Modeling Strategy

Bibliography of a personal quest

• B.N. and Dogbè C., On the modelling of traffic and crowds - a survey of models, speculations, and perspectives, *SIAM Review*, **53** (2011), 409–463.

• B.N. and Bellouquid A., On the modeling of crowd dynamics: Looking at the beautiful shapes of swarms, *Netw. Heter. Media*, **6** (2011), 383–399.

• B.N., Knopoff D., and Soler J., On the difficult interplay between life, “complexity”, and mathematical sciences, *Math. Models Methods Appl. Sci.*, **23** (10) (2013), 1861–1913.

• B.N., Bellouquid A., and Knopoff D., From the micro-scale to collective crowd dynamics, *SIAM Multiscale Model. Simul.*, **11** (2013), 943–963.

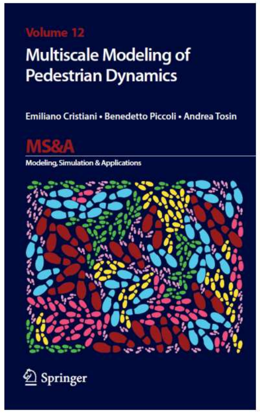
• B.N. and Bellouquid A., On multiscale models of pedestrian crowds - From mesoscopic to macroscopic, *Comm. Math. Sci.*, (2015), to appear.

• B.N. and L. Gibelli, Toward a mathematical theory of behavioral-social dynamics for pedestrian crowds, *arXiv:1411.0907v1*, (2014).

1.2 - From “What is a Crowd?” to a Modeling Strategy

Bibliography: Additional Readings

• E. Cristiani, B. Piccoli, and A. Tosin, **Multiscale Modeling of Pedestrian Dynamics**, Springer, (2014).



• R.L. Hughes, The flow of human crowds, *Annu. Rev. Fluid Mech.*, **35** (2003), 169–182.

• N. Bellomo, B. Piccoli, and A. Tosin, Modeling crowd dynamics from a complex system viewpoint, *Math. Models Methods Appl. Sci.*, **22** (2012), paper n. 1230004.

• L. Pareschi and G. Toscani, **Interacting Multiagent Systems: Kinetic equations and Monte Carlo methods**, Oxford University Press, USA, (2013).

1.4 - From “What is a Crowd?” to a Modeling Strategy

Why a crowd is a social, hence complex, system?

- **Ability to express a strategy:** Walkers are capable to develop specific strategies, which depend on their own state and on that of the entities in their surrounding environment.
- **Heterogeneity and hierarchy:** The ability to express a strategy is heterogeneously distributed and includes, in addition to different walking abilities, also different objectives and the possible presence of leaders.
- **Nonlinear interactions:** Interactions are nonlinearly additive and involve immediate neighbors, but also distant individuals.
- **Social communication and learning ability:** Walkers have the ability to learn from past experience. Therefore, their strategic ability evolves in time due to inputs received from outside induced by the tendency to adaptation.
- **Influence of environmental conditions:** The dynamics is remarkably affected by the quality of environment, including weather conditions, and the geometry of the domain.

1.5 - From “What is a Crowd?” to a Modeling Strategy

Complexity features of crowds - Definitions

• ***Definition of crowd:*** Agglomeration of many people in the same area at the same time. The density of people is assumed to be high enough to cause continuous interactions, or reactions, with other individuals.

• ***Collective intelligence:*** Emergent functional behavior of a large number of people that results from interactions of individuals rather than from individual reasoning or global optimization.

• ***Crowd turbulence:*** Unanticipated and unintended irregular motion of individuals into different directions due to strong and rapidly changing forces in crowds of extreme density.

• ***Emergence of spontaneous behaviors:*** Establishment of a qualitatively new behavior through non-linear interactions of many objects or subjects. In some cases it can be defined a ***Black Swan***.

1.6 - From "What is a Crowd?" to a Modeling Strategy

Complexity features of crowds - Definitions

• ***Faster-is-slower effect:*** Certain processes (in evacuation situations, production, traffic dynamics, or logistics) take more time if performed at high speed. In other words, waiting can often help to coordinate the activities of several competing units and to speed up the average progress.

• ***Freezing-by-heating effect:*** Noise-induced blockage effect caused by the breakdown of direction-segregated walking patterns (typically two or more lanes characterized by a uniform direction of motion). Noise means frequent variations of the walking direction due to nervousness or impatience in the crowd.

• ***Panic breakdown of ordered, cooperative behavior of individuals:*** Anxious reactions to a certain event. Often, panic is characterized by attempted escape of many individuals from a real or perceived threat in situations of a perceived struggle for survival, which may end up in trampling or crushing.

1.7 - Scaling Problems and Mathematical Structures

Levels of Description: Micro, Meso, Macro

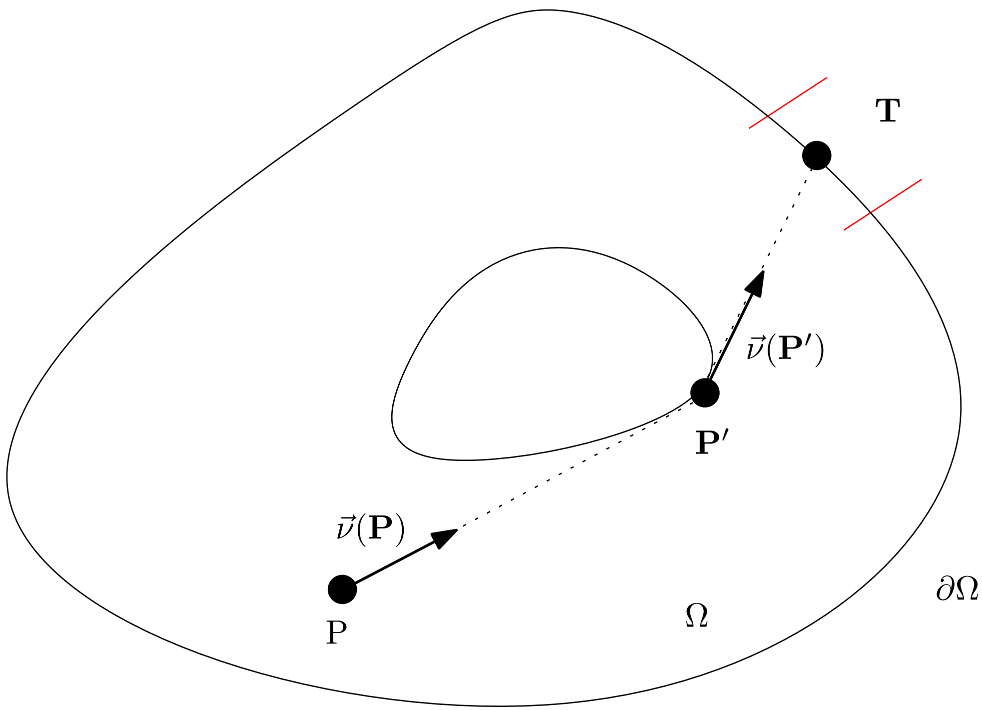
• ***Microscale:*** Walkers are individually identified. The state of the whole system is identified by their position and velocity, which dependent variables of time, . Mathematical models are generally stated by systems of ordinary differential equations.

• ***Mesoscale:*** The microscopic state of the interacting entities is still identified by the position and velocity, but their representation is delivered by a suitable probability distribution over the microscopic state. Mathematical models describe the evolution of the above distribution function generally by nonlinear integro-differential equations.

• ***Macroscale:*** The state of the system is described by averaged gross quantities, namely density and linear momentum, regarded as dependent variables of time and space. Mathematical models describe the evolution of the above variables by systems of partial differential equations.

1.8 - Scaling Problems and Mathematical Structures

Geometry



The set of all walls, including that of obstacles, is denoted by Σ.

1.10 - Scaling Problems and Mathematical Structures

Microscale

• The ***microscopic description*** of the pedestrian dynamics is represented, for each i-th walker with $i \in \{1, \dots, N\}$, by the following dimensionless variables: The position vector $\mathbf{x}_i = \mathbf{x}_i(t) = (x_i(t), y_i(t))$ and the velocity vector $\mathbf{v}_i = \mathbf{v}_i(t) = (v_x^i(t), v_y^i(t))$.

• Mathematical models are generally stated as a large system of ordinary differential equations where $\mathbf{x}_i$ and $\mathbf{v}_i$ are the dependent variables.

$$\begin{cases} \dfrac{d\mathbf{x}_i}{dt} = \mathbf{v}_i \, , \\ \\ \dfrac{d\mathbf{v}_i}{dt} = \mathbf{F}_i(\mathbf{x}_1, \dots, \mathbf{x}_N, \mathbf{v}_1, \dots, \mathbf{v}_N; \Sigma), \end{cases}$$

and where the dynamics depends also on geometry Σ including the inlet and outlet gates.

1.10. - Scaling Problems and Mathematical Structures

Mesoscale

• The overall systems can be subdivided into different groups of persons, called functional subsystems, which develop different strategies or express them in a different way.

• The approach of the so-called behavioral crowd dynamics introduces an additional microscopic variable $u \in [0, 1]$, which models the heterogeneous ability of people. Then the overall state of the system is described by the generalized one-particle distribution function

$$f_i = f_i(t, \mathbf{x}, \mathbf{v}, u) = f_i(t, \mathbf{w}) \;:\quad [0, T] \times \Omega \times D_{\mathbf{v}} \times D_u \;\to \mathbf{R}_+,$$

such that $f_i(t, \mathbf{x}, \mathbf{v}, u)\, d\mathbf{x}\, d\mathbf{v}\, du = f_i(t, \mathbf{w})\, d\mathbf{w}$ denotes the number of active particles whose state, at time t, is in the interval $[\mathbf{w}, \mathbf{w} + d\mathbf{w}]$ of the i-th subsystem, where $\mathbf{w} = \{\mathbf{x}, \mathbf{v}, u\}$ is an element of the space of the microscopic states.

$$\rho(t, \mathbf{x}) = \int_{D_{\mathbf{v}} \times D_u} f(t, \mathbf{x}, \mathbf{v}, u)\, d\mathbf{v}\, du\,, \qquad \mathbf{q}(t, \mathbf{x}) = \int_{D_{\mathbf{v}} \times D_u} \mathbf{v}\, f(t, \mathbf{x}, \mathbf{v}, u)\, d\mathbf{v}\, du\,,$$

1.11 - Scaling Problems and Mathematical Structures

Mathematical structures

• ***Macroscale:*** The *local density* $\rho = \rho(t, \mathbf{x})$ which is referred to the maximum density n_M of walkers; the *mean velocity* $\mathbf{V} = \mathbf{V}(t, \mathbf{x})$, which is referred to maximum mean velocity V_M of walkers.

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \, \mathbf{V}) = 0 \, , \\ \\ \partial_t \, \mathbf{V} + (\, \mathbf{V} \cdot \nabla_{\mathbf{x}}) \, \mathbf{V} = \mathcal{A}[\rho, \mathbf{V}; \Sigma] \, , \end{cases}$$

where $\mathcal{A}[\rho, \mathbf{V}; \Sigma]$ is a psycho-mechanical acceleration acting on walkers in the elementary macroscopic volume of the physical space. This acceleration depends also on Σ.

• First-order models are obtained by mass conservation only linked to a closure of the equilibrium velocity $\mathbf{V} \cong \mathbf{V}_e(\rho; \Sigma)$.

• Second-order models are obtained by both equations along with a phenomenological relation describing the psycho-mechanic acceleration $\mathcal{A}[\rho, \mathbf{V}; \mathbf{V}_e, \Sigma]$.

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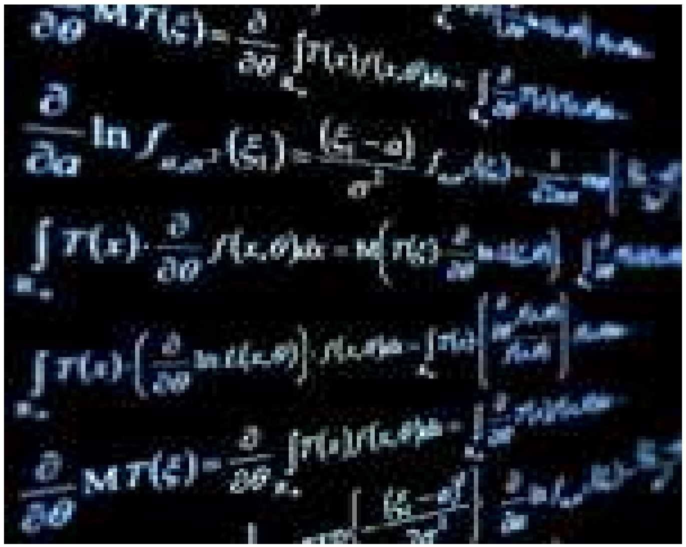
1.From the Question “What is a Crowd?” to a Modeling Strategy

2. On the Kinetic Theory Approach and Evacuation Dynamics



3. From Microscopic to Macroscopic

2.1 - On the Kinetic Theory Approach and Evacuation Dynamics

How mathematical sciences can contribute to understand the behavioral dynamics of crowds?

• B.N. and L. Gibelli, Toward a mathematical theory of behavioral-social dynamics for pedestrian crowds, *arXiv:1411.0907v1*, (2014).

- The overall system is subdivided into *functional subsystems* constituted by entities, called *active particles*, whose individual state is called *activity*;
- Each functional subsystem is featured by different ways of expressing their own strategy;
- The state of each functional subsystem is defined by a time dependent, probability distribution over the micro-scale state, which includes position, velocity, and activity;
- Interactions are modeled by games theory, more precisely stochastic games, where the state of the interacting particles and their outputs are known in probability;
- The evolution of the probability distribution is obtained by a balance of number particles within elementary volume of the space of the microscopic states, where the dynamics of inflow and outflow of particles is related to interactions.

2.2 - On the Kinetic Theory Approach and Evacuation Dynamics

Interactions by stochastic games: Living entities, at each interaction, *play a game* with an output that technically depends on their strategy somehow related to adaptation abilities. The output of the game generally is not deterministic and **the dynamics depends also on the overall shape of the walls including inlet and outlet doors**.

• **Test** particles of the i-th functional subsystem with microscopic state, at time t, delivered by the variable $(\mathbf{x}, \mathbf{v}, u) := \mathbf{w}$, whose distribution function is $f_i = f_i(t, \mathbf{x}, \mathbf{v}, u) = f_i(t, \mathbf{w})$. The test particle is assumed to be representative of the whole system.

• **Field** particles of the k-th functional subsystem with microscopic state, at time t, defined by the variable $(\mathbf{x}^*, \mathbf{v}^*, u^*) := \mathbf{w}^*$, whose distribution function is $f_k = f_k(t, \mathbf{x}^*, \mathbf{v}^*, u^*) = f_k(t, \mathbf{w}^*)$.

• **Candidate** particles, of the h-th functional subsystem, with microscopic state, at time t, defined by the variable $(\mathbf{x}_*, \mathbf{v}_*, u_*) := \mathbf{w}_*$, whose distribution function is $f_h = f_h(t, \mathbf{x}_*, \mathbf{v}_*, u_*) = f_h(t, \mathbf{w}_*)$.

2.3 - On the Kinetic Theory Approach and Evacuation Dynamics

Mathematical Structures of the Kinetic Theory for Active Particles

H.1. Candidate or test particles in $\mathbf{x}$, interact with the field particles in the interaction domain $\mathbf{x}^* \in \Omega$. Interactions are weighted by the ***interaction rate*** $\eta_{hk}[\mathbf{f}]$, which is supposed to depend on the local distribution function at the position of the field particles.

H.2. A candidate particle modifies its state according to the probability density: $\mathcal{C}^i_{hk}[\mathbf{f}](\mathbf{v}_* \to \mathbf{v},\, u_* \to u|\mathbf{w}_*, \mathbf{w})$, which denotes the probability density that a candidate particles of the h-subsystems with state $\mathbf{w}_* = \{\mathbf{x}_*, \mathbf{v}_*, u_*\}$ reaches the state $\{\mathbf{v}, u\}$ in the i-th subsystem after an interaction with the field particles of the k-subsystems with state $\mathbf{w}^* = \{\mathbf{x}^*, \mathbf{v}^*, u^*\}$.

Normalized **dimensionless** variables are used by dividing the number n of people per unit area with respect to the maximum number n_M corresponding to packing, and the velocity modulus (speed) v_r to the limit velocity v_ℓ

$$\rho = \frac{n}{n_M}, \qquad v = \frac{v_r}{v_\ell}.$$

2.4 - On the Kinetic Theory Approach and Evacuation Dynamics

Balance within the space of microscopic states and Structures

Variation rate of the number of active particles

$= $ *Inlet flux rate caused by number conservative interactions*

$-$*Outlet flux rate caused by conservative interactions*,

which corresponds to the following structure:

$$
\begin{aligned}
(\partial_t + \mathbf{v}\cdot\partial_{\mathbf{x}})\ f_i(t,\mathbf{x},\mathbf{v},u) &= \big(J_i^C - J_i^L + J_i^P - J_i^D\big)[\mathbf{f}](t,\mathbf{x},\mathbf{v},u) \\
&= \sum_{h,k=1}^{n} \int_{\Omega\times D_u^2\times D_{\mathbf{v}}^2} \eta_{hk}[\mathbf{f}](\mathbf{w}_*,\mathbf{w}^*)\,\mathcal{C}_{hk}^i[\mathbf{f}](\mathbf{v}_*\to\mathbf{v},\ u_*\to u|\mathbf{w}_*,\mathbf{w}^*,u_*) \\
&\quad\times f_h(t,\mathbf{x},\mathbf{v}_*,u_*) f_k(t,\mathbf{x}^*,\mathbf{v}^*,u^*)\,d\mathbf{v}_*\,d\mathbf{v}^*\,du_*\,du^*\,d\mathbf{x}^* \\
&\quad - \sum_{k=1}^{n} f_i(t,\mathbf{x},\mathbf{v}) \int_{\Omega\times D_u\times D_{\mathbf{v}}} \eta_{ik}[\mathbf{f}](\mathbf{w}_*,\mathbf{w}^*)\, f_k(t,\mathbf{x}^*,\mathbf{v}^*,u^*)\,d\mathbf{v}^*\,du^*\,d\mathbf{x}^*.
\end{aligned}
$$

Michail Gromov, In a Search for a Structure, Part 1: On Entropy, *Preprint*, (2013), http://www.ihes.fr/ gromov/.

2.5 - On the Kinetic Theory Approach and Evacuation Dynamics

Stochastic Games

1. **Competitive (dissent):** When one of the interacting particle increases its status by taking advantage of the other, obliging the latter to decrease it. Therefore the competition brings advantage to only one of the two. This type of interaction has the effect of increasing the difference between the states of interacting particles, due to a kind of driving back effect.

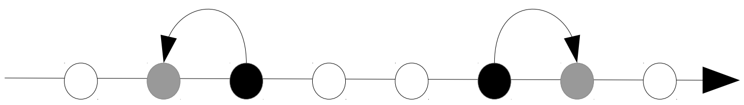
2. **Cooperative (consensus):** When the interacting particles exchange their status, one by increasing it and the other one by decreasing it. Therefore, the interacting active particles show a trend to share their micro-state. Such type of interaction leads to a decrease of the difference between the interacting particles' states, due to a sort of dragging effect.

3. **Hiding-chasing:** When one of the two attempts to increase the overall distance from the other, which attempts to reduce it.

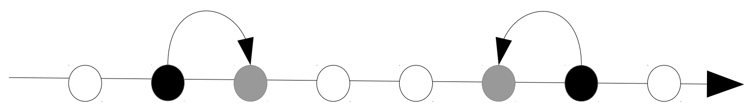
4. **Learning:** When one of the two modifies, independently from the other, the micro-state, in the sense that it learns by reducing the distance between them.

2.6 - On the Kinetic Theory Approach and Evacuation Dynamics

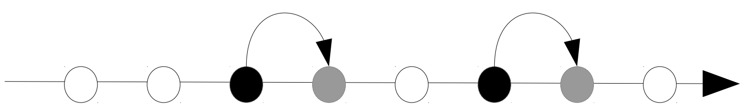
Stochastic Games - *Pictorial illustration of (a) competitive,(b) cooperative, (c) hiding-chasing and (d) learning game dynamics between two active particles. Black and grey bullets denote, respectively, the pre- and post-interaction states of the particles.*



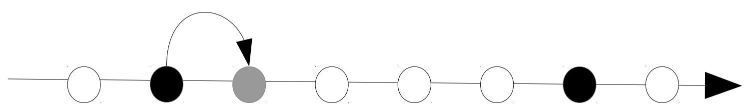
(a) Competition



(b) Cooperation



(c) Hiding-chasing



(d) Learning

2.7 - On the Kinetic Theory Approach and Evacuation Dynamics

Active particles, micro-scale states, and environment

Active particles	Walkers
Microscopic state	Position Velocity Activity
Functional subsystems	Different abilities Individuals pursuing different strategies Presence of leaders
Environment	Unbounded domains Domains with obstacles and boundaries Quality of the environment

2.8 - Models, Problems, and Evacuation Dynamics

Mesoscopic (kinetic) representation in polar coordinates

• The dynamics in two space dimensions is considered, while polar coordinates are used for the velocity variable, namely $\mathbf{v} = \{v, \theta\}$, where v is the velocity modulus and θ denotes the velocity direction.

• The ***perceived density*** ρ_θ^a along the direction θ:

$$\rho_\theta^a = \rho_\theta^a[\rho] = \rho + \frac{\partial_\theta \rho}{\sqrt{1 + (\partial_\theta \rho)^2}} \left[(1-\rho)\, H(\partial_\theta \rho) + \rho\, H(-\partial_\theta \rho)\right],$$

where ∂_θ denotes the derivative along the direction θ, while $H(\cdot)$ is the heaviside function $H(\cdot \geq 0) = 1$, and $H(\cdot < 0) = 0$. Therefore, positive gradients increase the perceived density up to the limit $\rho = 1$, while negative gradients decrease it down to the limit $\rho = 0$ in a way that

$$\partial_\theta \rho \to \infty \Rightarrow \rho^a \to 1\,, \quad \partial_\theta \rho = 0 \Rightarrow \rho^a = \rho\,, \quad \partial_\theta \rho \to -\infty \Rightarrow \rho^a \to 0.$$

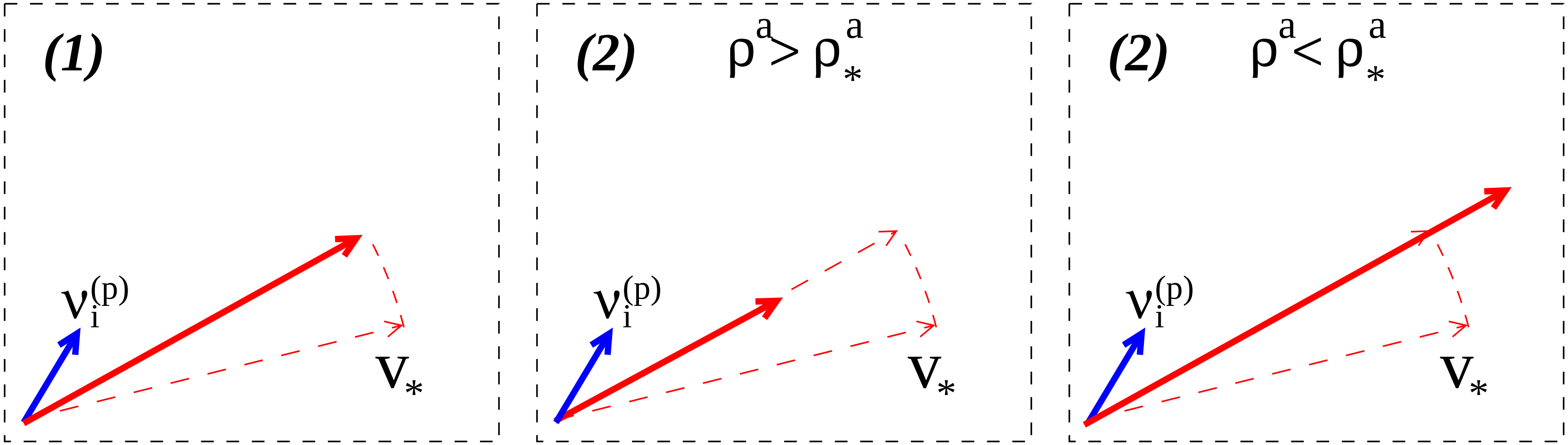
2.9 - On the Kinetic Theory Approach and Evacuation Dynamics

Modeling the decision process of velocity adjustment

1. In unbounded domains three types of stimuli contribute to the modification of walking direction: (i) desire to reach a well defined target, namely a direction or a meeting point; (ii) attraction toward the mean stream; (iii) attempt to avoid overcrowded areas (in domains with boundaries also the presence of walls induce an additional stimulus to avoid them).
2. Walkers moving from one direction to the other adapt their velocity to the new local perceived density conditions, namely they decrease speed for increasing perceived density and increase it for decreasing perceived density.
3. The activity variable according to a social dynamics based on attraction and/or repulsion of social behaviors.
4. The dynamics is more rapid in high quality areas; moreover rapidity is heterogeneously distributed and increases for high values of the activity variable.

2.10. On the Kinetic Theory Approach and Evacuation Dynamics

Dynamics at the microscopic scale



Interactions modify the dynamics of walkers in three steps:

1. direction of movement is changed depending on local density, mean velocity, and trend to the exit;
2. modulus of velocity is decreased (increased) depending on the perceived density;
3. activity variable is varied according to a social dynamics based on attraction and/or repulsion of social behaviors.

2.11. Models, Problems, and Evacuation Dynamics

- Three types of stimuli contribute to modify the walking direction:
  1. desire to reach a well defined target, $\boldsymbol{\nu}_i^{(t)}$;
  2. attraction toward the mean stream, $\boldsymbol{\nu}^{(s)}$;
  3. attempt to avoid overcrowded areas, $\boldsymbol{\nu}^{(v)}$.
- The preferred direction is defined by

$$\omega = \frac{(1-\rho)\boldsymbol{\nu}_i^{(t)} + \rho\left[\varepsilon\boldsymbol{\nu}^{(s)} + (1-\varepsilon)\boldsymbol{\nu}^{(v)}\right]}{\left\|(1-\rho)\boldsymbol{\nu}_i^{(t)} + \rho\left[\varepsilon\boldsymbol{\nu}^{(s)} + (1-\varepsilon)\boldsymbol{\nu}^{(v)}\right]\right\|}, \qquad \varepsilon \in [0,1],$$

  where

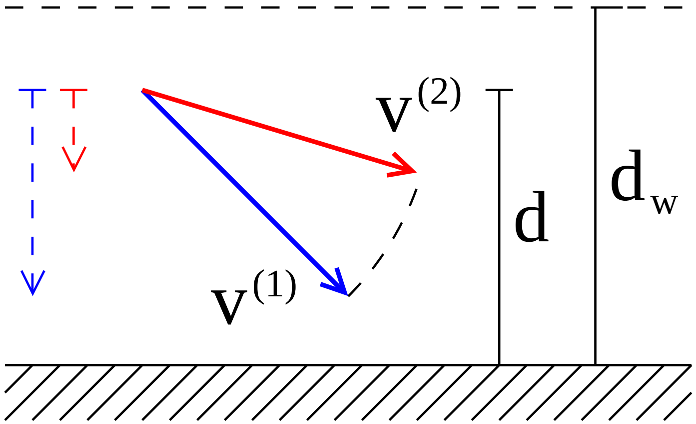
$$\boldsymbol{\nu}^{(s)} = \frac{\mathbf{V}}{\|\mathbf{V}\|}, \qquad \boldsymbol{\nu}^{(v)} = -\frac{\nabla\rho}{\|\nabla\rho\|},$$

  and the parameter ε accounts for panic conditions.

2.11. Models, Problems, and Evacuation Dynamics

A further adjustment in the presence of boundaries follows:

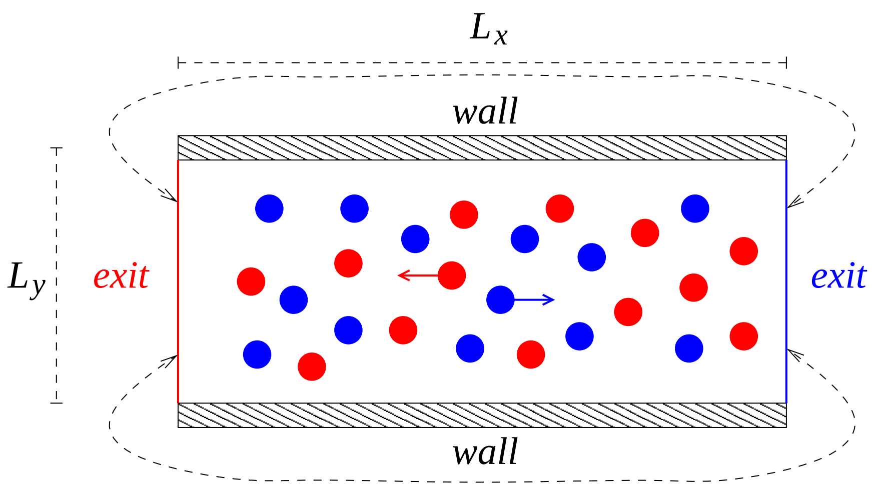
- The candidate walker changes in probability the direction of motion on the basis of the rules elaborated in unbounded domains;
- If its distance from the wall, d, is within a given cut-off, d_w, walker velocity is rotated so as the velocity component normal to the wall is decreased linearly with d.



$$
\begin{aligned}
v_n^{(2)} &= \frac{d}{d_w} v_n^{(1)} \\
v_t^{(2)} &= \text{sign}(v_t^{(1)}) \left[{v^{(1)}}^2 - {v_n^{(2)}}^2 \right]^{1/2}
\end{aligned}
$$

2.12. Models, Problems, and Evacuation Dynamics

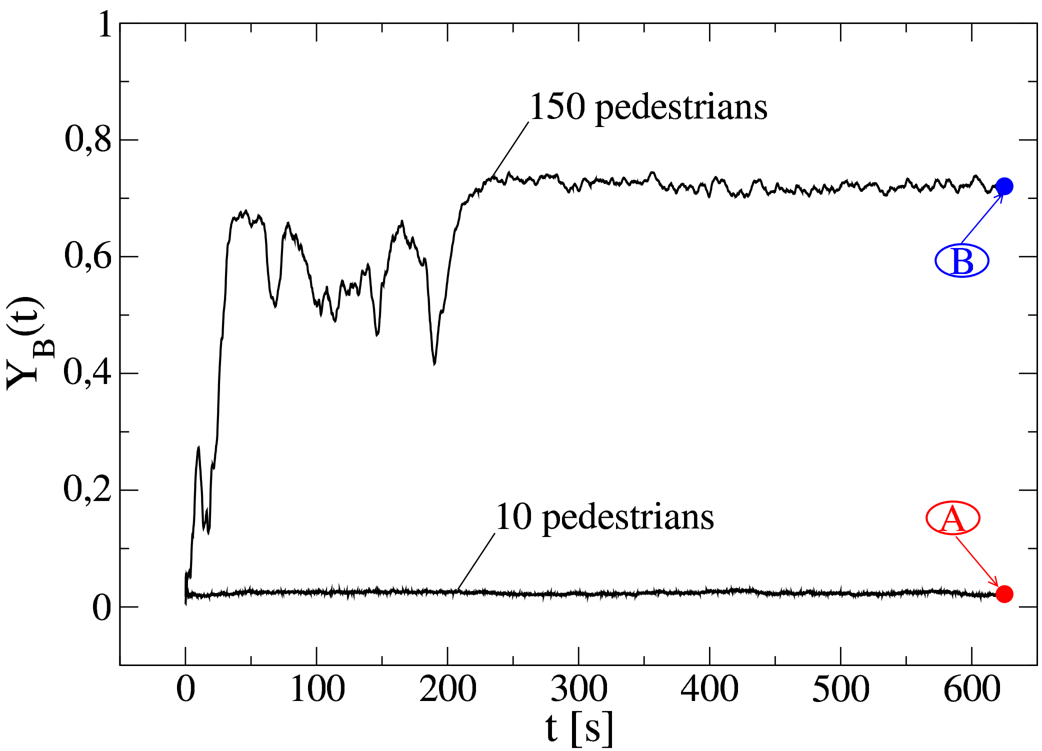
Individuals walking in a corridor with opposite directions



- The kinetic model of pedestrian crowds is applied to the problem of two groups of people walking in opposite directions.
- The segregation of walkers into lanes of uniform walking direction is quantitatively assess by computing the band index

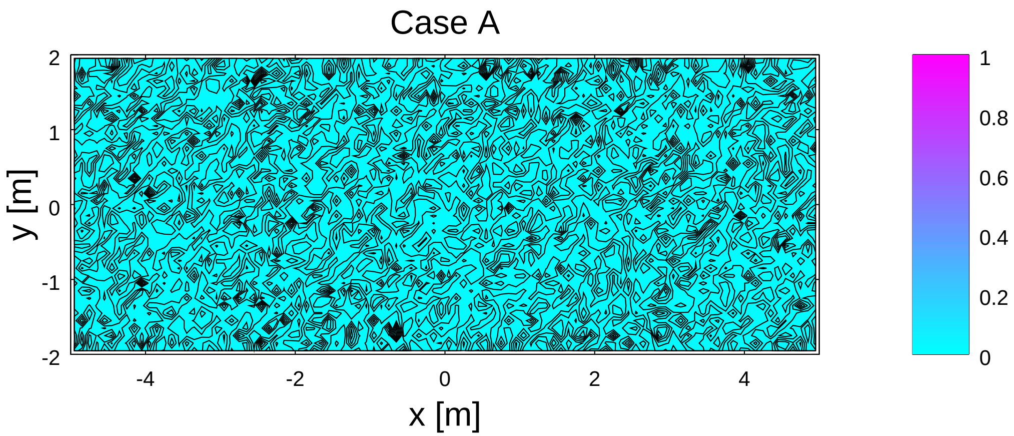
$$Y_{\mathrm{B}}(t) = \frac{1}{L_x L_y} \int_0^{L_y} \left| \int_0^{L_x} \frac{\rho_1(t,\mathbf{x}) - \rho_2(t,\mathbf{x})}{\rho_1(t,\mathbf{x}) + \rho_2(t,\mathbf{x})} dx \right| dy$$

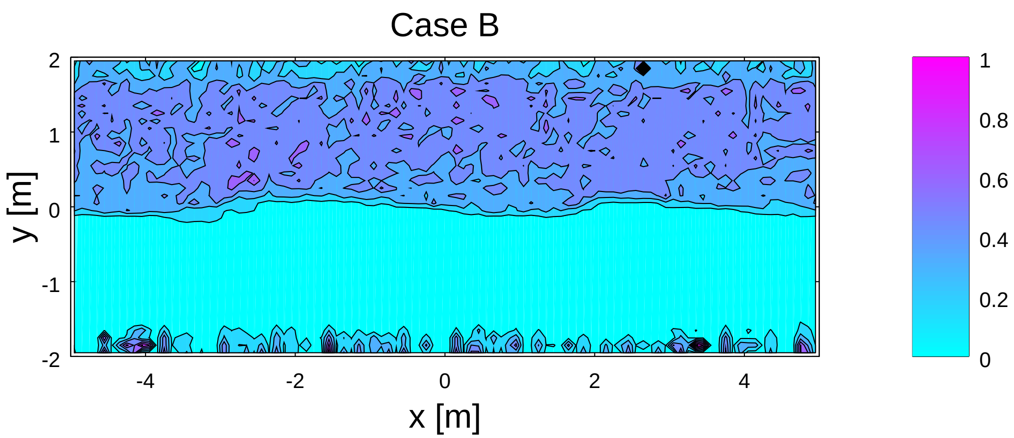
Pedestrians walking in a corridor with opposite directions



2.14. Models, Problems, and Evacuation Dynamics

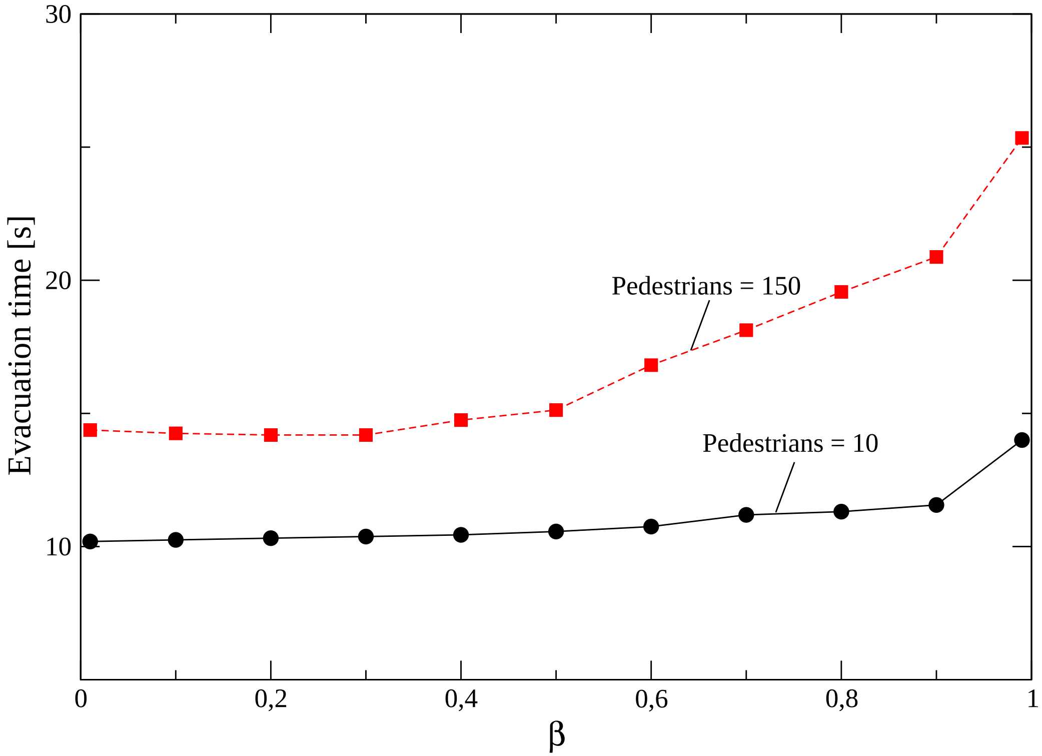
A: Low density flow; B: high density flow; $\varepsilon = .8$





2.15. Models, Problems, and Evacuation Dynamics

The role of the “selfishness” parameter Panic: *Breakdown of ordered, cooperative behavior of individuals due to anxious reactions to a certain event.* [Helbing D., Johansson A., (2009)]



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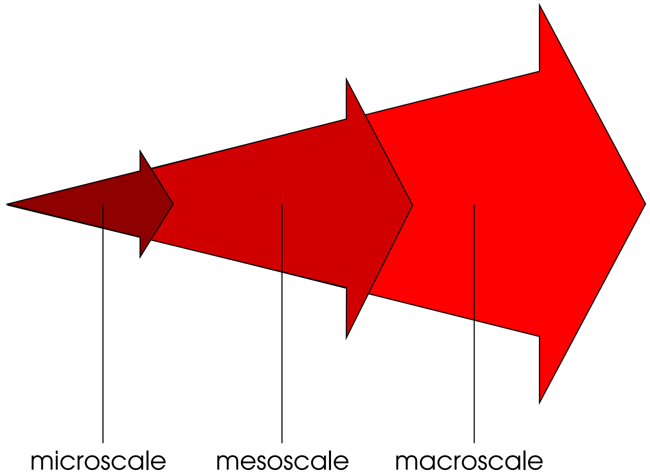

- 3.1 Using models of micro-scale dynamics to close macroscopic equations: mass conservation and linear momentum equation;
- 3.2 From micro-scale dynamics to kinetic type models and from kinetic type models to hydrodynamics.

3.1 - From Microscopic to Macroscopic

How multiscale problems can be treated?

From microscopic dynamics to hydrodynamics: mass conservation

Let us consider the mass conservation equation involving density and velocity depending on time and space coordinates, namely $\rho = \rho(t, \mathbf{x})$ and $\mathbf{V} = \mathbf{V}(t, \mathbf{x})$. Moreover, consider two phenomenological parameters:

- $\alpha \in [0, 1]$ which models the quality of the environment, where $\alpha = 1$ stands for the optimal quality of the environment, which allows to reach high velocity, while $\alpha = 0$ stands for the worst quality, which prevents the motion;
- $\varepsilon \in [0, 1[$ which models the attraction of walkers toward the direction of the mean velocity from $\varepsilon = 0$, which stands for highest search of less congested areas.

Two class of models can be studied:

- ***Homogeneous crowd***, where all walkers have the same walking ability;
- ***Heterogeneous crowd***, where walkers are subdivided in a number n of populations, labeled by the subscript i, corresponding to different functional subsystems

3.3 - From Microscopic to Macroscopic

From microscopic dynamics to hydrodynamics: Structure for mass conservation

- $\boldsymbol{Homogeneous\ \ crowd:}$ The mass conservation equation writes as follows:

$$\partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{V}) = 0.$$

The closure of the equation can be obtained by modeling the dependence of $\mathbf{V}$ on ρ by a phenomenological relation of the type $\mathbf{V} = \mathbf{V}[\rho](\alpha, \varepsilon)$, so that the conservation equation formally writes as follows:

$$\partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho\, \mathbf{V}[\rho](\alpha, \varepsilon)) = 0\,,$$

where square brackets denote that functional, rather than functions, relations can be used to link the local mean velocity to the local density.

Specific models can be obtained by heuristic interpretations of physical reality leading to $\mathbf{V} = \mathbf{V}[\rho]$ and inserting such models into the mathematical structure.

3.4 - From Microscopic to Macroscopic

From microscopic dynamics to hydrodynamics: Structure for mass conservation

Heterogeneous crowd: The state of the system is defined by a set of dimensionless number densities

$$\rho_i = \rho_i(t, \mathbf{x})\,, \qquad i = 1, \ldots, n\,, \qquad \rho(t, \mathbf{x}) = \sum_{i=1}^{n} \rho_i(t, \mathbf{x})\,,$$

where the subscripts correspond to a discrete variable, modeling the walking ability or different strategies.

The new structure simply needs the modification for a mixture:

$$\partial_t \rho_i + \nabla_{\mathbf{x}} \left(\rho_i \sum_{i=1}^{n} \beta\, V_i[\rho|\alpha, \varepsilon] \right) = 0 \qquad i = 1, \ldots, n\,,$$

where the modeling of the mean velocity differs for each population $\mathbf{V}_i = \mathbf{V}_i[\rho](\alpha, \varepsilon)$.

3.5 - From Microscopic to Macroscopic

Homogeneous crowd: Derivation of models requires simply to describe analytically the dependence of $\mathbf{V}$ on the local density distribution. Walkers first choose the preferred direction defined by the unit vector ω by taking into account the various stimuli defined in Part II. Subsequently they adapt their velocity modulus to the local perceived density $\rho^*[\rho]$: $V = V(\rho;\alpha)$, and according to the following constraints: $V(0) = \alpha$, $V'(0) = V(1) = V'(1) = 0$, where prime denotes derivative with respect to ρ.

The expression of ω has been already computed in Part II, while a model of V can be obtained from a simple polynomial approximation. Finally the model is as follows:

$$\partial_t \rho + \nabla_{\mathbf{x}} \Big(\rho\alpha(1 - 3\,\rho *^2 + 2\,\rho*^3)\omega(\rho^*, \varepsilon) \Big) = 0.$$

Heterogeneous crowd:

$$\partial_t \rho_i + \nabla_{\mathbf{x}} \Big(\rho_i \sum_{i=1}^{n} \frac{i}{n}\, \alpha(1 - 3\,\rho *^2 + 2\,\rho*^3)\omega_i(\rho^*, \varepsilon) \Big) \Big) = 0 \qquad i = 1, \ldots, n\,,$$

• B.N., S. Berrone, L. Gibelli, and A. Pieri, First Order Models of Crowds with Behavioral-Social Dynamics, *arXiv*, (2015).

3.6 - From Microscopic to Macroscopic

From Kinetic to Macroscopic

Step 1: Existence Calculations analogous to those we have seen in Part II, yield:

$$\begin{aligned}
\big[\partial_t \quad &+ \quad v_j(cos\theta_i \mathbf{i} + sin\theta_i \mathbf{j}) \cdot \nabla_{\mathbf{x}}\big] f_{ij}(t,\mathbf{x}) = \mathcal{J}_{ij}[\mathbf{f}](t,\mathbf{x}) \\
&= \quad \sum_{h,r=1}^{n} \sum_{k,s=1}^{m} \int_{\Omega[\mathbf{x}]} \eta(\rho^a(t,\mathbf{x}^*)) \mathcal{A}_{hk}^{rs}(ij)(\rho^a(t,\, \mathbf{x}^*); \alpha) f_{hk}(t,\mathbf{x})\, f_{rs}(t,\mathbf{x}^*)\, d\mathbf{x}^* \\
&- \quad f_{ij}(t,\mathbf{x}) \sum_{k=1}^{n} \sum_{s=1}^{m} \int_{\Omega[\mathbf{x}]} \eta(\rho^a(t,\mathbf{x}^*))\, f_{ks}(t,\mathbf{x}^*)\, d\mathbf{x}^*, \quad \mathbf{f} = \{f_{ij}\}.
\end{aligned}$$

Existence for arbitrarily large times has been proved in the Banach space $X_T = C([0,T], L^1_{M_{2n,2m}})$ of the matrix-valued functions $f = f(t,\mathbf{x}) : [0,T] \times \Omega \to M_{2n,2m}$ endowed with the norm

$$\| f \|_{X_T} = \sup_{t \in [0,\, T]} \| f \|_1 ,$$

$$L^1_{M_{2n,2m}} = \{ f = (f_{ij}) \in M_{2n,2m} : \| f \|_1 = \sum_{i=1}^{n} \sum_{j=1}^{m} \int_{\Omega} |\, f_{ij}(t,\mathbf{x}) \,|\; d\mathbf{x} < \infty \}.$$

3.7 - From Microscopic to Macroscopic

From Kinetic to Macroscopic

Step 2: Derivation of Continuity Equations Continuity equation is obtained by summing with respect to i and j:

$$\partial_t \rho(t,\mathbf{x}) + \nabla_{\mathbf{x}} \cdot (\rho\,\mathbf{q})(t,\mathbf{x}) = 0.$$

The momentum equation is obtained by multiplying by v_{ij} and summing with respect to i and j:

$$\partial_t(\rho\,\mathbf{q})(t,\mathbf{x}) + \nabla_{\mathbf{x}} \cdot \sum_{i=1}^{n}\sum_{j=1}^{m}(v_{ij} \otimes v_{ij} f_{ij}(t,\mathbf{x})) = \sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij}\mathcal{J}_{ij}[\mathbf{f}](t,\mathbf{x}),$$

where it is important to distinguish between the transport and the source term, and where $\nabla_{\mathbf{x}} \cdot \sum_{i=1}^{n}\sum_{j=1}^{m}(v_{ij} \otimes v_{ij} f_{ij}(t,\mathbf{x}))$ denotes the vector

$$\nabla_{\mathbf{x}} \cdot \sum_{i=1}^{n}\sum_{j=1}^{m}(v_{ij} \otimes v_{ij} f_{ij}(t,\mathbf{x})) = \left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{2}\partial_{x_k}(v_{ij}^{\ell} v_{ij}^{k} f_{ij}(t,\mathbf{x}))\right)_{\ell},$$

where $\ell = 1,2$ correspond to the dimension of the space variable.

3.8 - From Microscopic to Macroscopic

From Kinetic to Macroscopic Step 3: Closure

Technical calculations yield:

$$\partial_t(\rho\,\mathbf{q})(t,\mathbf{x}) + \nabla_{\mathbf{x}} \cdot (P(t,\mathbf{x}) + \rho\,\mathbf{q} \otimes \mathbf{q})(t,\mathbf{x}) + E(t,\mathbf{x}) = S(t,\mathbf{x}),$$

where P, E and S are given by

$$\begin{aligned}
P(t,\mathbf{x}) &= \left(\sum_{i=1}^{n} \sum_{j=1}^{m} (v_{ij}^{(k)} - \mathbf{q}^{(k)}(t,\mathbf{x}))(v_{ij}^{(\ell)} - \mathbf{q}^{(\ell)}(t,\mathbf{x})) f_{ij}(t,\mathbf{x}) \right)_{1 \le k,\ell \le 2}, \\
E(t,\mathbf{x}) &= \sum_{i=1}^{n} \sum_{j=1}^{m} v_{ij} \left(\mid \Omega[\mathbf{x}] \mid \mathcal{J}_{ij}^{\star}[\mathbf{f}](t,\mathbf{x}) - \mathcal{J}_{ij}[\mathbf{f}](t,\mathbf{x}) \right), \\
S(t,\mathbf{x}) &= \mid \Omega[\mathbf{x}] \mid \sum_{i=1}^{n} \sum_{j=1}^{m} v_{ij} \mathcal{J}_{ij}^{\star}[\mathbf{f}](t,\mathbf{x}).
\end{aligned}$$

Closure follows by maximum entropy calculations.

• B.N. and Bellouquid A., On multiscale models of pedestrian crowds - From mesoscopic to macroscopic, *Comm. Math. Sci.*, (2015), to appear.

3.9 - From Microscopic to Macroscopic

An "first order closure" can be obtained from the solution of the stationary homogeneous case, which is denoted by $f_{ij}^e(\rho)$. Accordingly, if the stationary distribution is known, the *equilibrium quantities* can also be defined, namely, the mean equilibrium velocity becomes:

$$\mathbf{q}_e(\rho)(t,\mathbf{x}) = \frac{1}{\rho(t,\mathbf{x})} \sum_{i=1}^{n} \sum_{j=1}^{m} v_{ij} f_{ij}^e(\rho)(t,\mathbf{x}),$$

while the equilibrium pressure is:

$$P^e(\rho)(t,\mathbf{x}) = \sum_{i=1}^{n} \sum_{j=1}^{m} (v_{ij} - \mathbf{q}_e(\rho)(t,\mathbf{x})) \otimes (v_{ij} - \mathbf{q}_e(\rho)(t,\mathbf{x})) f_{ij}^e(\rho)(t,\mathbf{x}).$$

The term E, can be estimated by using Taylor formula as follows

$$E = E^e[\mathbf{f}] \sim -V \cdot \nabla_{\mathbf{x}} \rho \Big(a_1(\rho) + \rho\, \mathbf{q}_e(\rho)\, a_2(\rho) \Big) + O(|\, \Omega[\mathbf{x}]\, |),$$

and the vector V is given by

$$V = \int_{\Omega[\mathbf{x}]} (\mathbf{x}^* - \mathbf{x}) d\mathbf{x}^*.$$

3.10 - From Microscopic to Macroscopic

"A second order closure" can be obtained by approximation of the density and momentum equations when $| \Omega[\mathbf{x}] | \to 0$ produces the following system:

$$\begin{cases} \partial_t \rho(t,x) + \nabla_x \cdot (\rho\, \mathbf{q})(t,x) = 0\,, \\ \\ \partial_t(\mathbf{q})(t,x) + \frac{1}{\rho}\, \nabla_{\mathbf{x}} \cdot P^e(\rho) + \mathbf{q} \cdot \nabla_{\mathbf{x}} \mathbf{q} + d(\rho) V \cdot \nabla_{\mathbf{x}} \rho = \mu(\rho)\, (\mathbf{q}_e(\rho) - \mathbf{q}), \end{cases}$$

where the vector $\mathbf{q} \cdot \nabla_{\mathbf{x}} \mathbf{q}$ and the coefficients $d(\rho)$ and $\mu(\rho)$are given by:

$$\mathbf{q} \cdot \nabla_{\mathbf{x}} \mathbf{q} = \left(\sum_{k=1}^{2} (\partial_{\mathbf{x}_k} q_\ell) \mathbf{q}_k \right)_{1 \le \ell \le 2},$$

$$d(\rho) = -\frac{1}{\rho} \Big(a_1(\rho) + \rho\, \mathbf{q}_e(\rho)\, a_2(\rho) \Big), \quad \mu(\rho) = | \, \Omega[\mathbf{x}] \, | \, \nu(\rho).$$

For technical details see

• B.N. and Bellouquid A., On multiscale models of pedestrian crowds - From mesoscopic to macroscopic, *Comm. Math. Sci.*, (2015), to appear.

3.11 - Open Problems

Open Problems and Perspectives

Which are the most challenging research perspectives?

- **Modeling a variety of not usual behaviors and computing their propagation in space.**
- **Modeling evacuation dynamics in complex venues including passages from one area to an other.**
- **Qualitative analysis of the initial-boundary value problems for the dynamics in domains with boundaries.**
- **Derivation of macroscopic models from the underlying description at the micro-scale for dynamics in domains with boundaries.**
- **Others ?**

Complexity can be fascinating and **Thank You!**