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Toward a mathematical theory of behavioral-social dynamics for pedestrian crowds

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This paper presents a new approach to behavioral-social dynamics for pedestrian crowds by suitable development of mathematical tools of the kinetic theory. It is shown how pedestrians heterogeneity and the propagation of local unusual behaviors in the crowd can be accounted for. The proposed model is applied to the study of two groups of pedestrians walking in opposite directions in a crowded street and its predictive ability is demonstrated by showing that emerging behaviors, such as pedestrian segregation, can be depicted.

Keywords: Scaling; kinetic theory; active particles; crowd dynamics; pedestrian segregation.

1. Plan of the Paper

This paper deals with the modeling of *behavioral-social crowd dynamics*, where this term is used to indicate that laws of classical mechanics can be substantially modified by individual behaviors and strategies developed by living entities. As a matter of fact, individuals in a crowd can be viewed as self-propelled particles, which are featured by a heterogeneously distributed ability to express their walking ability and strategy related to the interactions with the other entities.

The paper aims at presenting a unified approach, to modeling and simulation, based on theoretical tools of the kinetic theory and stochastic games,[11] which, according to the authors' bias, offer an appropriate framework to capture the

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greatest part of the complexity features of the systems under consideration. Classical methods of the kinetic theory for molecular fluids cannot be straightforwardly applied. Indeed, mass conservation can be claimed in the case of self-propelled particles, but not conservation of momentum and energy. The modeling involves several technical difficulties whose understanding is a necessary preliminary step to develop a successful approach.

The interested reader can take advantage of some review papers, which enlighten different aspects of the modeling of crowd dynamics. Restricting our attention to the most recent literature, the survey paper[12] presents and critically analyses a variety of crowd models at all representation scales, from the microscopic (individual) to macroscopic (hydrodynamical). A mathematical approach to the modeling of swarm dynamics, which has some intersection with crowd dynamics, is developed in Ref. 13.

Reference 7 is proposed for crowd dynamics in unbounded domains with homogeneous distribution of walking ability of pedestrians. This paper aims at including more general features of behavioral-social dynamics, as well as interactions with walls. Moreover, the approach of Ref. 7 uses discrete velocity variables, while a continuous variable is used in this paper. This choice allows one to overcome the uncertainty problem induced by the choice, up to now heuristic, of the number of velocity modulus and directions. The contents of the paper are as follows.

Section 2 defines an assessment of the basic principles of *behavioral-social dynamics* of a large system of interacting self-propelled particles with focus on crowd dynamics. The analysis is related to the complexity features of these specific systems and anticipates the mathematical kinetic theory for active particles.[11] The microscopic state of the particles, which constitutes the micro-scale system, includes not only mechanical variables, typically position and velocity, but also an additional variable, called activity, suitable to model their strategy.

Section 3 develops the concept of behavioral-social dynamics as a new approach to dynamics, where interactions depend on the behaviors of the interacting entities viewed as a living system. A general structure is designed toward the derivation of specific models. The approach consists in representing the state of the system by a probability distribution over the micro-state of particles and in deriving a general framework to describe the time and space dynamics of such distribution by a balance of particles in the elementary volume of the space of the micro-states. Interactions, which are nonlocal and nonlinearly additive, are modeled by theoretical tools of stochastic game theory.

Section 4 takes advantage of the framework presented in the preceding section to derive models which also include the dynamics of individuals which change the rules of their participation to the dynamics. The whole section is structured in three parts. First the dynamics is modeled in unbounded domains. Then it is shown how the decision process by which individuals select their trajectories include the actions to avoid walls and obstacles. Finally, it is shown how their social behavior evolves in time and space focusing on panic conditions.[21,23]

Section 5 presents some simulations with the aim to test the predictive ability of the model. A particle method of solution is used as this approach appears to be well consistent with the general mathematical structure proposed in Sec. 3. Two types of simulations are developed, the first one shows how the individuals moving in opposite directions interact, while the second one aims at understanding how panic conditions can affect the evacuation dynamics. Reference 22 is an important reference to define specific objectives of simulations.

Section 6 presents a critical analysis and indicates how the approach can be further developed to include additional aspects of the dynamics such as modeling interactions which induce large deviations in the social behavior of the crowd.

2. Behavioral-Social Dynamics of Self-Propelled Particles

Let us now understand what is a crowd and which are the most important features to be taken into account in the modeling approach. First a definition of the *crowd* needs to be given. The following (due to Helbing and Johansson[22]) appears to be particularly well focused:

> *Agglomeration of many people in the same area at the same time. The density of pedestrians is assumed to be high enough to cause continuous interactions with or reaction to other individuals.*

According to the authors' bias, only three important specific features can be selected among various ones. Subsequently, a critical analysis of the existing literature will focus on topics to be considered open and to be treated in the next sections.

• *Strategy*: Pedestrians have the ability to express walking strategies based on interactions with other individuals and with the surrounding environment. The latter includes vocal and visual signallers which address them toward optimal and safe ways, including evacuation paths. *The modeling of pedestrians' strategy should include several features, for instance trend toward the exit or a meeting point, following or avoiding streams and clusters, avoiding overcrowding in the proximity of walls, clustering of individuals with similar activity, avoiding individuals with different activity, and possibly others.*

• *Heterogeneity*: The behavior of pedestrians is heterogeneously distributed due to both different psychologic attitudes and mobility abilities from individuals with handicap to high level walking ability. In addition, a crowd might need to be split into different groups related to different strategies, e.g. reaching different objectives, or even an internal hierarchy, which induces different interaction rules. *Heterogeneity can refer to social behaviors, for instance aggressiveness in a crowd where two groups contrast each other or panicking behaviors. Indeed, crowds can lose, in panic*

conditions, optimal strategies. This feature can be important in extreme situations when sudden dangers can induce the said conditions.

• *Interactions*: Interactions involve both mechanical and social-behavioral features, and are nonlocal as individuals communicate and develop a visual activity at a distance. Moreover, they are nonlinearly additive as the strategy developed by a pedestrian is a nonlinear combination of different stimuli, while mechanics induces social exchanges which, in turn, modify the walking dynamics. *Interactions with the external environment where the pedestrians move, namely different geometrical and environmental contexts (corridors, rooms, stairs, sudden changes of directions, luminosity conditions, and so on) can have an important effect on the dynamics since the interaction rules depend on the quality of the environment.*

Although only three general features have been selected, one can rapidly verify that the existing literature does not yet provide an exhaustive answer to all of them. In fact, recent papers, e.g. Refs. 1 and 7, only treat some of the aforesaid topics, while the interplay between mechanics and social behaviors deserves further attention. The meso-scale approach, introduced in Ref. 6 and further developed in Ref. 1, describe pedestrians as heterogeneous entities which move according to a decision process modeled by tools of game theory.

It is an approach quite different from that developed at the microscopic scale, for instance by the social force model,[20] which is based on the assumption that pedestrians are subject to an acceleration to be related to social interactions within the crowd. However, models at the microscopic scale can contribute not only to understand how individual behaviors can be described by equations, but also to model interactions. A useful example is given by the paper of Faure and Maury,[18] which provides a detailed analysis of the granular dynamics based on different aspects of attraction and repulsion between pedestrians.

The next sections present a model which should at least partially include the predictive ability of the aforesaid features. The derivation is followed by a critical analysis of the said ability also referred to validation issues. This analysis can take advantage of the literature in the field, which offers valuable contributions on theoretical models and interpretation of empirical data, among others.[15,25,26,29,30]

Finally, let us mention the strategic motivation to study anomalous behaviors and, in particular, the onset and propagation of panic conditions. This study is motivated by the related safety problems, see Refs. 21 and 23 for a deeper understanding of a psycho-mechanical study of this extreme phenomenon and Ref. 1 for a detailed analysis of evacuation times and the impact that panic conditions can have on it. Indeed, the study of this feature is an important issue of this paper.

3. On the Kinetic Theory Approach to Behavioral Dynamics

Let us consider a large heterogeneous system of pedestrians moving in two-dimensional domains. The mathematical approach to modeling cannot relay, as

observed in Ref. 11, on the deterministic causality principles typical of classical mechanics. In fact, pedestrians develop their own dynamics based on an individual interpretation of that of the other individuals. As already mentioned, they develop a strategy, which is heterogeneously distributed and which depends on several factors to be included in the modeling approach. These reasonings lead to introduce the already-mentioned concept of *behavioral-social dynamics.* This section proposes a general structure to be used toward the derivation of specific models.

Bearing this in mind, let us anticipate some terminology and some preliminary ideas of the approach that will be developed hereinafter.

- The modeling approach proposed in this paper is based on suitable developments of the so-called kinetic theory for active particles, which applies to large systems of interacting entities.[11] Hence the *meso-scale* representation is chosen.
- Pedestrians, namely the *micro-system*, are viewed as *active particles*, that have the ability of expressing their own strategy, called *activity.* This ability can differ for different groups in the same crowd, being understood that the activity is heterogeneously distributed.
- Pedestrians can communicate and develop a *social dynamics*, as they learn from interactions and accordingly modify both strategy and dynamical rules followed in the movement. The output is a collective behavior which can be observed in the whole.
- The overall system is subdivided into groups of pedestrians who share common features according to the hallmarks of a systems theory of social systems introduced in Ref. 2 and followed in Refs. 10 and 17. We refer to these groups as *functional subsystems.*
- Each functional system is described by a *probability distribution over the microscopic state of pedestrians*, namely of the variable deemed to define their physical state, while interactions are modeled by theoretical tools of game theory.[27]

A general overview of this approach is presented in the survey paper,[11] while applications to model crowd dynamics and social systems are proposed in Refs. 6 and 7. The modeling is developed in three steps, each of them presented in the next subsections, namely representation of the system, modeling of micro-scale interactions and derivation of a mathematical structure consistent with the complexity paradigms of behavioral dynamics. This last subsection also proposes a concise critical analysis.

3.1. *Representation*

Let us now consider a crowd which can be subdivided into different groups distinguished either by different walking objectives and/or abilities. Each group, called *functional subsystem*, has a different walking purpose, for instance either following a certain direction or reaching a meeting point, or even different roles in the crowd corresponding, for instance, to hierarchical behaviors.

The *microscopic state* of pedestrians, viewed as *active particles*, is defined by position $\mathbf{x}$, velocity $\mathbf{v}$, and activity u. Dynamics in two-space dimensions is considered, while polar coordinates are used for the velocity variable, namely $\mathbf{v} = \{v, \theta\}$, where v is the velocity modulus and θ denotes the velocity direction.

The ability of pedestrians to express the strategy of the group to which they belong is modeled by a variable $u \in [0, 1]$, called *activity*, such that $u = 0$ denotes the worst walking ability, while $u = 1$ the best one.

Dimensionless, or normalized, quantities are used by referring the components x and y to a characteristic length ℓ, while the velocity modulus is divided by the limit velocity, V_ℓ, which can be reached by a fast pedestrian in free flow conditions; t is the dimensionless time variable obtained referring the real time to a suitable reference time T_r identified by the ratio between ℓ and V_ℓ. The limit velocity depends on the quality of the environment, say presence of positive or negative slopes, lighting and so on. It should be measured both in normal and panic flow conditions as it will be discussed later. For the case study considered in this work, we assume that $\ell = 1\,\mathrm{m}$ and $V_\ell = 1.6\,\mathrm{m/s}$ in the best quality environments.

The *mesoscopic* (*kinetic*) *representation* of the overall system is delivered by the statistical distribution at time t, over the micro-scale state:

$$f_i = f_i(t, \mathbf{x}, v, \theta, u), \quad v \in [0, 1], \quad \theta \in [0, 2\pi) \quad u \in [0, 1], \tag{3.1}$$

for each functional subsystem labeled by $i = 1, \ldots, n$.

If f_i is locally integrable, then $f_i(t, \mathbf{x}, v, \theta, u) v\, dv\, d\theta\, du\, d\mathbf{x}$ is the (expected) infinitesimal number of pedestrians whose micro-state, at time t, is comprised in the elementary volume of the space of the micro-states, corresponding to the variables space, velocity and activity, of each functional subsystem. The statistical distributions are normalized by n_M, which defines the maximal full packing density of pedestrians and it is assumed to be $7\,\mathrm{ped/m}^2$.

The following compact expression is occasionally used in the following:

$$\mathbf{w} = \{v, \theta, u\}, \quad \text{with} \quad \mathbf{w} \in D_{\mathbf{w}} = [0, 1] \times [0, 2\pi) \times [0, 1],$$

so that $f_i = f_i(t, \mathbf{x}, v, \theta, u) = f_i(t, \mathbf{x}, \mathbf{w})$.

Macroscopic observable quantities can be obtained, under suitable integrability assumptions, by weighted moments of the distribution functions. As an example, the density and mean velocity of pedestrians belonging to the functional subsystem i read

$$\rho_i(t, \mathbf{x}) = \int_0^1 \int_0^{2\pi} \int_0^1 f_i(t, \mathbf{x}, v, \theta, u) v\, dv\, d\theta\, du \tag{3.2}$$

and

$$\boldsymbol{\xi}_i(t, \mathbf{x}) = \frac{1}{\rho_i(t, \mathbf{x})} \int_0^1 \int_0^{2\pi} \int_0^1 \mathbf{v} f_i(t, \mathbf{x}, v, \theta, u) v\, dv\, d\theta\, du, \tag{3.3}$$

whereas global expressions are obtained by summing over the index labeling the functional subsystems

$$\rho(t,\mathbf{x}) = \sum_{i=1}^{n} \rho_i(t,\mathbf{x}) \quad \text{and} \quad \boldsymbol{\xi}(t,\mathbf{x}) = \frac{1}{\rho(t,\mathbf{x})} \sum_{i=1}^{n} \rho_i(t,\mathbf{x})\boldsymbol{\xi}_i(t,\mathbf{x}). \tag{3.4}$$

An additional quantity to be taken into account in the modeling of interactions is the *perceived density* ρ_θ^a along the direction θ. According to Ref. 6, this quantity is defined as follows:

$$\rho_\theta^a = \rho_\theta^a[\rho] = \rho + \frac{\partial_\theta \rho}{\sqrt{1+(\partial_\theta\rho)^2}}[(1-\rho)H(\partial_\theta\rho) + \rho H(-\partial_\theta\rho)], \tag{3.5}$$

where ∂_θ denotes the derivative along the direction θ, while $H(\cdot)$ is the Heaviside function, $H(\cdot \geq 0) = 1$ and $H(\cdot < 0) = 0$. Therefore, positive gradients increase the perceived density up to the limit $\rho = 1$, while negative gradients decrease it down to the limit $\rho = 0$ in a way that

$$\partial_\theta\rho \to \infty \Rightarrow \rho^a \to 1, \quad \partial_\theta\rho = 0 \Rightarrow \rho^a = \rho, \quad \partial_\theta\rho \to -\infty \Rightarrow \rho^a \to 0.$$

3.2. *Hallmarks toward modeling interactions*

As already mentioned, interactions may depend not only on the micro-state of the interacting particles (pedestrians), but also on their distribution functions. When only the first type of interaction appears, one talks about *linear interactions*, otherwise, when also second type occurs, the concept of *nonlinear interactions* needs to be used. Linearity involves only independent variables, i.e. micro-state variables, while nonlinearity involves the dependent variables, i.e. the distribution function and/or its moments. In the following, round and square parenthesis distinguish the former and latter interactions, respectively.

The modeling of interactions corresponds to a decision process for each particle related to the micro-state, and the distribution function of the particle and of all those in its interaction domain. For each functional subsystem, three types of particles are involved in the process at each time t:

- The *test particle*, in $\mathbf{x}$, with micro-state $\mathbf{w}$ and distribution function $f_i(t,\mathbf{x},\mathbf{w})$.
- The *field particle*, in $\mathbf{x}^*$, with micro-state $\mathbf{w}^*$ and distribution function $f_i(t, \mathbf{x}^*, \mathbf{w}^*)$.
- The *candidate particle*, in $\mathbf{x}$, with micro-state $\mathbf{w}_*$ and distribution function $f_i(t,\mathbf{x},\mathbf{w}_*)$.

The candidate particle can acquire, in probability, the micro-state of the test particle after interaction with the field particles, while the test particle loses its state in the interaction with the field particles. The test particle is representative, for each functional subsystem, of the whole system of particles.

Interactions of test and candidate particles with field particles, can be modeled by the following quantities: *interaction domain* Ω, *interaction rate* η, and *transition*

probability density $\mathcal{A}$. These quantities can depend, as already mentioned, on the micro-state and distribution function of the interacting particles, as well as on the quality of the environment. Moreover, they refer to interactions involving all functional subsystems. The formal expression of these terms is reported in the following, where the term i-particle is used to denote a pedestrian belonging to the ith functional subsystem.

- *Interaction domain*: Active particles have an interaction domain Ω to be related to their visibility domain which can be defined as a circular sector, with radius R, symmetric with respect to the velocity direction being defined by the visibility angles Θ and $-\Theta$.
- *Interaction rate*: This term models the frequency by which a candidate h-particle (or test) in $\mathbf{x}$ develops contact with a field k-particle in the visibility zone, $\mathbf{x}^* \in \Omega$. The following notation, referred to candidate and field particles, can be used $\eta_{hk}[f](\mathbf{w}_*, \mathbf{w}^*; \alpha)$.
- *Transition probability density*: Interactions can be modeled by the probability density $\mathcal{A}^i_{hk}[f](\mathbf{w}_* \to \mathbf{w} \,|\, \mathbf{w}_*, \mathbf{w}^*; \alpha)$, which gives the probability that a candidate h-particle with state $\mathbf{w}_*$ in $\mathbf{x}$ becomes an i-particle with state $\mathbf{w}$ due to the interaction with a field k-particle in $\mathbf{x}^* \in \Omega$ with state $\mathbf{w}^*$. When the system does not allow transitions across functional subsystems, the following notation is used $\mathcal{A}^i_{ik} =: \mathcal{A}_{ik}$, while when interactions do not depend on the other subsystems one has $\mathcal{A}^i_{ii} =: \mathcal{A}_i$.
- *Role of the parameter* α: In general, both η_{hk} and $\mathcal{A}^i_{hk}$ can depend on the micro-scale states of the interacting particles and on the density of the field particles in the domain Ω. However, interactions also depend on the quality of the environment which can be modeled by the parameter $\alpha \in [0, 1]$, where $\alpha = 0$ correspond to the worse conditions preventing the dynamics, while $\alpha = 1$ to the best ones, which allow a rapid dynamics.

3.3. *A mathematical structure*

The hallmarks to derive models by the kinetic theory for active particles, which we also follow in this paper, are reported in Ref. 11. In detail, first a general structure suitable to capture the main complexity features of living systems is derived and subsequently such a structure is implemented by models of interactions at the micro-scale. This approach aims at overcoming the lack of first principles that govern the living matter. The mathematical framework consists in an integro-differential equation suitable to describe the time dynamics of the distribution functions f_i, which can be obtained by a balance of particles in the elementary volume of the space of the micro-states. This conservation law writes:

$$\textbf{Variation rate of the number of active particles} \\ = \textbf{Inlet flux rate} - \textbf{Outlet flux rate},$$

where the inlet and outlet fluxes are caused by interactions. This equality corresponds to the following structure:

$$(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}}) f_i(t, \mathbf{x}, \mathbf{w}) = J_i[f](t, \mathbf{x}, \mathbf{w}; \alpha) = \eta_0 (G_i - L_i)[f](t, \mathbf{x}, \mathbf{w}; \alpha), \quad (3.6)$$

where η_0 is a scaling parameter which accounts for normalization.

When the crossing over functional subsystems, induced by change of behaviors, is allowed, the interaction term of the structure reads

$$G_i = \sum_{h=1}^{n} \sum_{k=1}^{n} \int_{\Omega \times D_{\mathbf{w}}^2} \eta_{hk}[f](\mathbf{w}_*, \mathbf{w}^*; \alpha)\, \mathcal{A}_{hk}^i[f](\mathbf{w}_* \to \mathbf{w} \,|\, \mathbf{w}_*, \mathbf{w}^*; \alpha)$$
$$\times f_h(t, \mathbf{x}, \mathbf{w}_*) f_k(t, \mathbf{x}^*, \mathbf{w}^*) d\mathbf{w}_*\, d\mathbf{w}^*\, d\mathbf{x}^* \quad (3.7)$$

and

$$L_i = f_i(t, \mathbf{x}, \mathbf{w}) \sum_{k=1}^{n} \int_{\Omega \times D_{\mathbf{w}}} \eta_{ik}[f](\mathbf{w}, \mathbf{w}^*; \alpha) f_k(t, \mathbf{x}^*, \mathbf{w}^*) d\mathbf{w}^*\, d\mathbf{x}^*. \quad (3.8)$$

In the case of individuals who do not move from one functional subsystem to the other, the gain term, Eq. (3.7), simplifies to

$$G_i = \sum_{k=1}^{n} \int_{\Omega \times D_{\mathbf{w}}^2} \eta_{ik}[f](\mathbf{w}_*, \mathbf{w}^*; \alpha)\, \mathcal{A}_{ik}[f](\mathbf{w}_* \to \mathbf{w} \,|\, \mathbf{w}_*, \mathbf{w}^*; \alpha)$$
$$\times f_i(t, \mathbf{x}, \mathbf{w}_*) f_k(t, \mathbf{x}^*, \mathbf{w}^*) d\mathbf{w}_*\, d\mathbf{w}^*\, d\mathbf{x}^*. \quad (3.9)$$

Notice should be made that this structure is consistent with the paradigms presented in Sec. 2. The ability of pedestrians to express walking strategies based on interactions with other individuals is modeled by the transition probability density, while the heterogeneous distribution of the said strategy (behavior) corresponding both to different psychologic attitudes and mobility abilities is taken into account by the use of a probability distribution over the activity variable. Interactions have been assumed to be nonlocal and nonlinearly additive as the strategy developed by a pedestrian is a nonlinear combination of different stimuli generated by interactions with other pedestrians and with the external environment.

4. Mathematical Models

Specific models are hereinafter derived within the mathematical framework proposed in the preceding section. Attention is restricted to crowd dynamics when pedestrians have a well-defined strategic objective and do not cross from one functional subsystem to the other. In this specific case, the activity variable corresponds to the walking ability and the derivation of models needs defining the interaction rate, η_{ik}, and the transition probability density, $\mathcal{A}_{ik}$, which describes interactions at the micro-scale. In general, the interaction rate can depend on the micro-state and the meso-state of the interacting individuals viewed as active particles. A rule generally adopted is that the frequency of interactions depends on the number of

particles in the interaction domain. However, the conjecture that individuals consider only a fixed number of individuals to develop the decision process on their walking strategy was posed in Ref. 5. See also the general formalization of Ref. 13 and the computational hints in Ref. 3. Therefore, if the visibility allows to capture this critical density, a reasonable assumption is to regard the interaction rate as a constant. However, lack of visibility or presence of obstacles can reduce this interaction rate.

The next two subsections refer to the modeling of the transition probability density and develop models in bounded and unbounded space domains, the third one focuses on the modeling of panic conditions and finally the last subsection is devoted to a critical analysis on the predictive ability of the model.

4.1. *Models in unbounded space domains*

Firstly some phenomenological assumptions on the qualitative micro-scale dynamics are made and, subsequently, these are transferred into models to be implemented into the mathematical structure. In detail, interactions modify the dynamics of pedestrians who first change the direction of movement, then the velocity modulus and finally the value of their activity. More specifically:

(i) Three types of stimuli contribute to the modification of walking direction:
  (a) desire to reach a defined target, namely a direction or a meeting point;
  (b) attraction toward the mean stream;
  (c) attempt to avoid overcrowded areas.

(ii) Pedestrians moving from one direction to the other, adapt their velocity to the new local perceived density conditions, namely they decrease speed for increasing perceived density and increase it for decreasing perceived density.

(iii) The activity variable is modified according to a social dynamics based on attraction and/or repulsion of social behaviors.

The dynamics is more rapid in high quality areas, namely for greater value of α. Let us consider a candidate pedestrian located in $\mathbf{x}$ with velocity direction θ_* and modulus v_*, and let us define the following unit vectors:

- $\boldsymbol{\nu}_i^{(\text{target})}(\mathbf{x})$ directed to a prescribed meeting point or to a walking direction;
- $\boldsymbol{\nu}_i^{(\text{stream})}[f](\mathbf{x})$ along the local stream which we assume to be defined by the mean velocity;
- $\boldsymbol{\nu}^{(\text{vacuum})}[f](\mathbf{x})$ directed along the direction of lowest density gradient.

Pedestrians in addition to the trend to $\boldsymbol{\nu}_i^{(\text{target})}$, which depends on the position only, develop a decision process between the attraction by the stream, namely $\boldsymbol{\nu}_i^{(\text{stream})}$, and the search of a low density path $\boldsymbol{\nu}^{(\text{vacuum})}$, where both unit vectors depend on $\mathbf{x}$ and on f through the mean velocity and mean density computed within the visibility zone, respectively. Reference 1 suggests that a preferred direction can be heuristically chosen. Accepting this hint, the following model of *preferred*

direction is proposed:

$$\boldsymbol{\nu}_i^{(p)} = \frac{(1-\rho)\boldsymbol{\nu}_i^{(\mathrm{target})} + \rho\left[(1-\beta)\boldsymbol{\nu}^{(\mathrm{vacuum})} + \beta\boldsymbol{\nu}_i^{(\mathrm{stream})}\right]}{\left\|(1-\rho)\boldsymbol{\nu}_i^{(\mathrm{target})} + \rho\left[(1-\beta)\boldsymbol{\nu}^{(\mathrm{vacuum})} + \beta\boldsymbol{\nu}_i^{(\mathrm{stream})}\right]\right\|}, \tag{4.1}$$

where β is a parameter which models the sensitivity to the stream with respect to the search of vacuum.

Remark 4.1. A correspondence can be easily identified between the preferred direction $\boldsymbol{\nu}_i^{(p)}$ and the preferred angle of motion, $\theta_i^{(p)}$, i.e. $\boldsymbol{\nu}_i^{(p)} = (\cos\theta_i^{(p)}, \sin\theta_i^{(p)})$.

All tools have now been defined to model the decision process leading to the transition probability density, $\mathcal{A}_i$, which is defined as follows:

$$\mathcal{A}_i[f](t,\mathbf{x},\mathbf{w};\alpha) = \mathcal{D}(u_* \to u)\mathcal{C}[\rho](v_* \to v;\alpha)\mathcal{B}_i[\rho,\boldsymbol{\xi}](\theta_* \to \theta), \tag{4.2}$$

where, according to Sec. 3.2, $\mathcal{A}_i$ stands for $\mathcal{A}_{ii}^i$.

More in detail:

(1) The candidate pedestrian in $\mathbf{x}$ with velocity direction θ_* changes in probability θ_* into θ depending on the direction $\theta_i^{(p)}$, on the local density, activity, and quality of the environment;
(2) After having changed velocity direction, the velocity is modified by increasing (decreasing) its modulus depending on whether the perceived density is lower (higher) than the previous one;
(3) Finally, the candidate pedestrian modifies the activity by social communications.

According to the aforesaid phenomenological description, the transition probability density for angles is assumed to vary linearly from θ_* to $\theta_i^{(p)}$, hence

$$\mathcal{B}_i[\rho,\boldsymbol{\xi}](\theta_* \to \theta = (\theta_* + \Delta\theta) \bmod 2\pi) = 2\frac{\phi_\theta - \psi_{i,\theta}}{\Delta\theta_{i,\max}^2}|\Delta\theta| + \frac{1+\psi_{i,\theta}-\phi_\theta}{|\Delta\theta_{i,\max}|}, \tag{4.3}$$

where $\Delta\theta \in [0, \Delta\theta_{i,\max}]$ and the maximum possible variation of velocity direction is given by

$$\Delta\theta_{i,\max} = (\theta_i^{(p)} - \theta_*) - 2\pi\,\mathrm{sign}(\theta_i^{(p)} - \theta_*)H(|\theta_i^{(p)} - \theta_*| - \pi), \tag{4.4}$$

where $H(\cdot)$ is the Heaviside function. In Eq. (4.3), ϕ_θ and ψ_θ give the negative and positive contributions to the trend to $\theta_i^{(p)}$, respectively

$$\phi_\theta = \rho \quad \text{and} \quad \psi_{i,\theta} = \alpha u_* \frac{|\Delta\theta_{i,\max}|}{\pi}. \tag{4.5}$$

Likewise, the transition probability density for the velocity modulus is given by

$$\mathcal{C}[\rho](v_* \to v = v_* + \Delta v) = 2\frac{\phi_v - \psi_v}{\Delta v_{\max}^2}|\Delta v| + \frac{1+\psi_v - \phi_v}{|\Delta v_{\max}|}, \tag{4.6}$$

where $\Delta v \in [0, \Delta v_{\max}]$ with $\Delta v_{\max} = v_p - v_*$ and the preferred velocity is given by $v_p = H(\rho_{\theta_*}^a - \rho_\theta^a)$. In Eq. (4.6), ϕ_θ and ψ_θ give the negative and positive contributions to the trend to v_p, respectively,

$$\phi_v = \rho \quad \text{and} \quad \psi_v = \alpha u_* \Delta v_{\max}. \tag{4.7}$$

Finally let us consider the third step consisting in modeling how interactions modify the activity variable, but still assuming that each pedestrian keeps one's own strategy without transition into another functional subsystem. The simplest assumption is that given a random initial condition of the distribution over such variable, then this is not modified by interactions. This assumption corresponds to the following expression of the transition probability density

$$\mathcal{D}(u_* \to u) = \delta(u - u_*). \tag{4.8}$$

4.2. *Models in bounded space domains*

When dealing with dynamics in a bounded domain, nonlocal interactions occur with walls, obstacles, and one or more exists. The strategy pursued in our modeling approach is that the aforementioned interactions, which replace the classical local boundary conditions, define a preferred walking direction in a two-step procedure. As a first step, the candidate pedestrian changes in probability the direction of motion and the velocity modulus by following the same rules described in the preceding subsection, Eqs. (4.3) and (4.6), respectively. As a second step, by keeping the same velocity modulus, the direction of motion is further changed so as to account for the presence of solid walls. Accordingly, the probability transition density modifies to

$$\begin{aligned} &\mathcal{A}_i[f](t, \mathbf{x}, \mathbf{w}; \alpha) \\ &\quad = \mathcal{B}^{(2)}(\theta^{(1)} \to \theta)\mathcal{D}[\rho](u_* \to u)\mathcal{C}[\rho](v_* \to v; \alpha)\mathcal{B}_i^{(1)}[\rho, \boldsymbol{\xi}](\theta_* \to \theta^{(1)}), \end{aligned} \tag{4.9}$$

where $\mathcal{B}_i^{(1)}$ is defined by Eq. (4.3), with $\theta^{(1)}$ in place of θ, and $\mathcal{B}^{(2)}$ reads

$$\mathcal{B}^{(2)}(\theta^{(1)} \to \theta) = \delta(\theta - \theta^{(2)}(\theta^{(1)})). \tag{4.10}$$

In Eq. (4.10), the angle of motion $\theta^{(2)}$ is obtained by rotating the velocity so as to reduce its normal component linearly with the distance from the wall.

More specifically, pedestrians whose distance from the wall, d, is within a cutoff, d_w, modify the components of their velocity as follows:

$$v_n^{(2)} = \frac{d}{d_w} v_n^{(1)}, \tag{4.11}$$

$$v_t^{(2)} = \mathrm{sign}\left(v_t^{(1)}\right)\left[{v^{(1)}}^2 - {v_n^{(2)}}^2\right]^{1/2}, \tag{4.12}$$

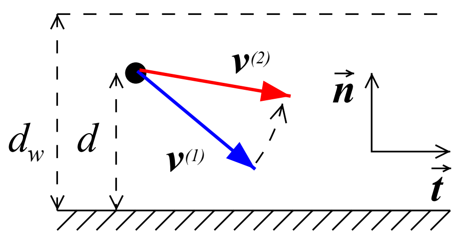

Fig. 1. Velocity adjustment in the presence of a wall.

where $v_t^{(1)}$ and $v_n^{(1)}$ are the tangential and normal velocity components of the pedestrian after the decisional process that weights target, stream and vacuum effects, respectively (see Fig. 1 for a pictorial representation). Equation (4.11) shows that the velocity component normal to the wall decreases linearly approaching the wall and becomes naught in the limiting case of $d = 0$, while the tangential component, as given by Eq. (4.12), is obtained by requiring that the modulus of the velocity is constant. The angle $\theta^{(2)}$, which enters in Eq. (4.10), is thus given by

$$\mathbf{v}^{(2)} = v_n^{(2)}\mathbf{n} + v_t^{(2)}\mathbf{t} = v^{(2)}(\cos\theta^{(2)}, \sin\theta^{(2)}), \tag{4.13}$$

where $\mathbf{n}$ and $\mathbf{t}$ are the unit vectors normal and tangential to the wall.

4.3. *Models in panic conditions*

As Ref. 1 already pointed out, in the presence of panic conditions pedestrians do not equally share the different trends but neglect the search of less congested areas and try to do what the others do. This feature is modeled by the parameter β in Eq. (4.1) which weights the relative importance of the attraction toward the stream. The panic also increases the walking ability, measured by the activity variable u, up to a certain extent. Besides these two effects, increasing the level of panic leads to an increase of the walking velocity of pedestrians as a whole. Therefore the limit velocity, which in normal conditions was indicated by V_ℓ, has to be substituted by a new limit velocity $V_L > V_\ell$.

If the crowd is in a sufficiently small domain, one can assume that at a certain critical time t_c the aforesaid effects are homogeneously captured by the whole population. Therefore the simplest approach is the following:

$$\begin{aligned} t \le t_c : \beta = \beta_1, \quad u = u_1, \\ t > t_c : \beta = \beta_2, \quad u = u_2, \end{aligned} \tag{4.14}$$

where $0 < \beta_1 < \beta_2 < 1$ and $0 < u_1 < u_2 < 1$. On the other hand, in case of overcrowding in a large environment, one can figure out situations where panic, or any anomalous behavior, is initially localized in a small area and is then transported in the domain. The modeling approach can be developed by adding β to the set which defines the micro-state, i.e. $\mathbf{w} = \{v, \theta, u, \beta\}$. Therefore, the transition probability density modifies as follows:

$$\begin{aligned} \mathcal{A}_i[f](t, \mathbf{x}, \mathbf{w}; \alpha) = \mathcal{E}[\rho](\beta_* \to \beta)\mathcal{D}(u_* \to u) \\ \times \mathcal{C}[\rho](v_* \to v; \alpha)\mathcal{B}_i[\rho, \boldsymbol{\xi}_i](\theta_* \to \theta), \end{aligned} \tag{4.15}$$

where now β is allowed to be modified by interactions. The modeling of $\mathcal{E}$ can be based on a consensus and learning type games, see Sec. 3 of Ref. 11 and the model follows by inserting this expression into the mathematical structure. As an example, $\mathcal{E}$ may be related to the distance of the states of two interacting particles, see Refs. 10 and 17.

Table 1. Pedestrian dynamics and related parameters.

Pedestrian dynamics	Parameter
Interactions among pedestrians.	η_0: Interaction rate, i.e. number per unit of time and space of pedestrians that change their velocity.
Three stimuli modify the velocity direction: (1) desire to reach a well-defined target; (2) attraction toward the mean stream; (3) attempt to avoid overcrowded areas.	β: Panic parameter, which weights relative importance of following the stream with respect to the search of vacuum.
Increasing (decreasing) **perceived density** decreases (increases) the velocity modulus.	α: Venue quality, which accounts for positive or negative slopes, good or bad lighting conditions and so on.
Presence of walls adjusts the velocity direction by reducing its normal component linearly with the distance from the wall.	d_w: Cutoff distance, i.e. distance from the wall below which the presence of solid walls has to be accounted for.

4.4. *Critical analysis*

The mathematical model proposed in the preceding subsections has been derived based on the idea that pedestrians interact with the other pedestrians as well as with the overall environment. These interactions are nonlocal and nonlinearly additive. Out of them each pedestrian selects a walking direction and adapts the velocity modulus to the perceived density conditions.

The main features of the dynamics are summarized in Table 1, along with the parameters entering in the sequential steps of the individual decision process which determines the velocity dynamics. In addition to these parameters, also V_ℓ and V_L have to be considered. As mentioned in Sec. 4.3, V_ℓ is the limit velocity in normal flow conditions, which can be individually reached by a fast pedestrian, whereas V_L is the limit velocity in panic conditions when all pedestrians collectively increase their velocity.

It is important to emphasize that both V_ℓ and V_L are related to α. Indeed the limit velocity depends on the quality of the venue, which can be featured by positive or negative slopes, as well as by good or bad lighting conditions and various others. The following simple relation is thus suggested:

$$V_L = \mu V_\ell = \mu\alpha V_M, \quad \mu > 1, \tag{4.16}$$

where V_M is the maximum speed in the best conceivable environment, namely flat and well lighted. Hence α and μ can be related to velocity measurements.

These reasonings do not claim to exhaustively cover the complexity of the experimental investigation on crowd dynamics[29,30] but are simply mentioned as perspectives for further investigation.

An additional criticism to be brought to the reader's attention is the modeling of panic conditions during evacuations. Some conjectures have been proposed in this field. These already provide, by simulations, some interesting results presented

in the next section. However, further research activity is needed as we shall see in the last section where some hints are proposed towards a more detailed analysis of interactions involving the activity variable, namely the so-called social interactions.

5. Simulations

Simulations are carried out to show the capability of the model to depict emerging behaviors which are observed in reality and are of interest in the evacuation dynamics. More specifically, we consider the problem of two groups of people walking in opposite directions in a crowded street.[24] The analysis focuses in particular on the role of the parameter β which is descriptive of the presence of panic conditions. It is well understood that this case study does not cover the whole varieties of dynamics to be considered. However, it leads to a number of dynamical behaviors worth to be studied. Hereinafter, we give a brief outline of the numerical method used in the present study to solve the kinetic model and subsequently we discuss the simulations results.

Obtaining numerical solutions of kinetic equations is a challenging task because the distribution function depends on a high number of variables and the computation of the interaction term requires the approximate evaluation of a multidimensional integral. Numerical approaches can be roughly divided into two broad categories, namely stochastic and deterministic. The basic idea of stochastic methods of solution consists in representing the distribution function by a number of computational particles which move in the space domain and interact according to stochastic rules derived from the kinetic equation.[14,16,28] The macroscopic fields are obtained through weighted averages of the particle properties. Deterministic methods of solution adopt a completely different strategy. They discretize the distribution function on a regular grid in the phase space. The streaming term is approximated by finite volume schemes and deterministic integration quadratures are used to evaluate the interaction integral.[4,19]

In most cases, kinetic equations for pedestrian dynamics are solved by means of deterministic methods.[1,7] By contrast, in the present work, we adopt a Monte Carlo particle scheme.[16,28] Compared with deterministic methods of solutions, the proposed approach provides some advantages such as the possibilities to easily account for complex geometries as well as to deal with sophisticated individual decision processes.

As mentioned, simulations are carried out for two groups of pedestrians moving in opposite directions in a narrow street. Figure 2 shows the geometry of the problem. Initially, the two groups of pedestrians are supposed to be uniformly distributed in the domain. The objective of the simulations is twofold:

(1) Describing the segregation of pedestrians into lanes of uniform walking direction;
(2) Assessing the influence of the parameter β and of the density ρ on the collective behavior.

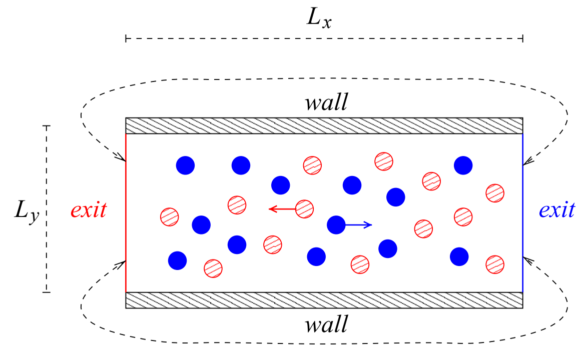

Fig. 2. Geometry of the case study: Street of dimension $L_x \times L_y = 10\,\mathrm{m} \times 4\,\mathrm{m}$. Periodic boundary conditions are assumed in the longitudinal direction.

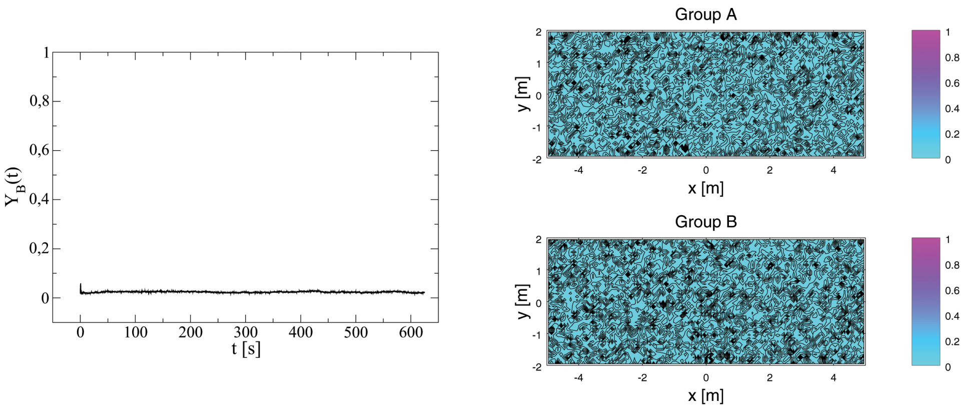

Fig. 3. Bidirectional flow of 10 pedestrians for $\beta = 0.8$. Left panel: Temporal evolution of the band index. Right panels: Density contour plots of the two groups of pedestrians at time $t = 625\,\mathrm{s}$.

The spontaneous formation of parallel lanes can be quantitatively assessed by computing the band index, $Y_{\mathrm{B}}(t)$, which measures the segregation of opposite flow directions.[31] In the present work it may be generalized to the expression

$$Y_{\mathrm{B}}(t) = \frac{1}{L_x L_y} \int_0^{L_x} \int_0^{L_y} \left| \frac{\rho_1(t, \mathbf{x}) - \rho_2(t, \mathbf{x})}{\rho_1(t, \mathbf{x}) + \rho_2(t, \mathbf{x})} \right| dx\, dy. \tag{5.1}$$

According to its definition, one has $Y_{\mathrm{B}}(t) = 0$ for mixed counterflows and $Y_{\mathrm{B}}(t) = 1$ for a perfect segregation of the opposite flows. As shown in Fig. 3, in a low density crowd, the band index, which is initially zero, does not change significantly in time. Indeed pedestrians randomly fill the domain and no segregation takes place. By contrast, the band index increases significantly for the case of a dense crowd reported in Fig. 4. The emergence of spatial segregation is apparent, with the two groups of pedestrians that form in line and file alternatively. It is worth pointing out that, unlike most of the previous studies on the subject, this emerging behavior has

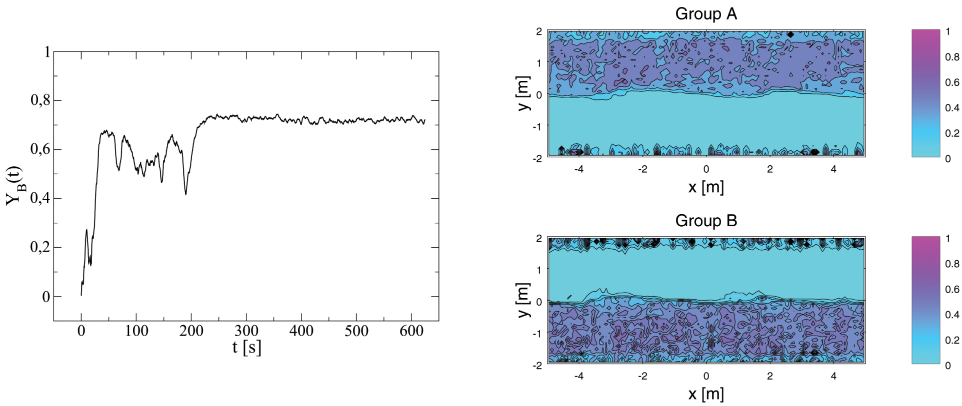

Fig. 4. Bidirectional flow of 150 pedestrians for $\beta = 0.8$. Left panel: Temporal evolution of the band index. Right panels: Density contour plots of the two groups of pedestrians at time $t = 625\,\mathrm{s}$.

not been obtained by introducing a repulsion force between pedestrians belonging to different populations.

Although the organization of pedestrians in lanes of uniform walking directions permits to enhance the efficiency of their movement, overall the mean flow rate decreases with the density of the crowd. This is clearly shown in Fig. 5 where the ratio between the pedestrian mean flow rate through a section of the street and the number of pedestrians is shown as a function of the number of pedestrians for different β. A threshold value, independent of β, can be found of about 125 pedestrians which corresponds to the mean crowd density of $\bar{\rho} \cong 0.5$. Below this threshold, the reduced mean flow rate is almost constant, whereas it strongly reduces for a higher number of pedestrians. This behavior is not unexpected since when $\bar{\rho} \gtrsim 0.5$ the

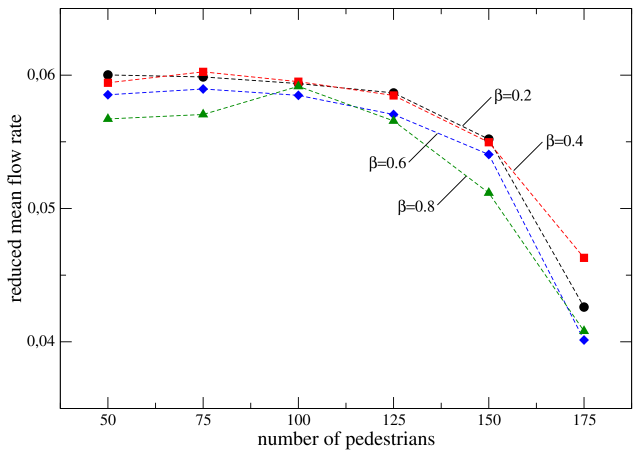

Fig. 5. Ratio between the pedestrian mean flow rate through a section of the street and the number of pedestrians versus the number of pedestrians for different β.

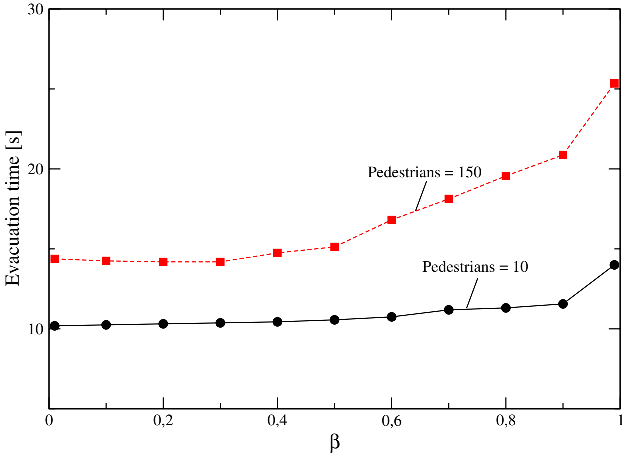

Fig. 6. Evacuation time for different numbers of pedestrians versus β.

stream and vacuum effects become dominant and this leads to a loss of efficiency in reaching the target.

Moreover, whatever is the number of pedestrians, the higher is the attraction to what the others do against the search of less congested areas, the lower is the mean flow rate. This is an aspect of the so-called faster-is-slower effect.[22] Such an effect can be further highlighted by evaluating the evacuation time of the two groups of pedestrians. This has been defined as the time they need to leave the domain when initially at rest and outflow conditions are used at the right and left boundaries instead of periodic boundary conditions. As shown in Fig. 6, the evacuation time increases with β, as much as the initial crowd density is greater. Indeed, a strong tendency of following other pedestrians leads to higher level of congestion which, in turn, determines a reduction of the pedestrians' speed.

6. Further Steps Toward a Social Crowd Dynamics

The hallmarks followed in this paper toward the objective of developing an approach to behavioral social dynamics can be listed as follows:

(1) Definition of the main features of behavioral dynamics;
(2) Derivation of a general structure suitable to offer the conceptual basis for the derivation of models;
(3) Modeling interactions at the microscopic scale to implement the aforesaid structure and derive specific models in unbounded and bounded space domains.

This process has been focused on a detailed analysis of the influence of panic conditions on the overall dynamics, which included some aspects of behavioral dynamics by the activity variable assumed to be heterogeneously distributed over individuals. It has been shown how this distribution evolves in time and space

due to interactions. However, the approach does not yet tackle a challenging problem, which consists in modeling the dynamics of a crowd subdivided into different functional subsystems behaving with different features and purposes such that the dynamics predicts transitions from one subsystem to the other.

It is worth mentioning that this type of investigation is motivated by security problems. As an example, one can consider a crowd of individuals in a public protest demonstration to support or make opposition to political issues. The crowd can be subdivided into a large group of individuals which manifest correctly their position, while a small group are rioters. The number of the latter can grow in time due to interactions which might persuade the other part of the crowd to join them. Similarly a small group of security forces might react to provocations out of the settled protocol, but their number might also grow due to interactions, which cause an excess of reactions. The modeling can take advantage of the more general structure given in Eqs. (3.7) and (3.8), as well as of the literature on social dynamics.[10,17]

This challenging program is not treated in this paper, however, some hints are given in the following as future research perspectives:

(1) The transition probability density should account for interactions which modify the activity variable according to theoretical tools of game theory as shown in Ref. 11. For instance, interactions can split the crowd into two groups, namely citizens involved in a calm protest and rioters.
(2) The transition from one functional subsystem to the other should be properly modeled. For instance, initially calm citizens increase their attraction to the riot and move into the group of rioters by a rate which increases with increasing activity variable.
(3) Simulations should be developed in the case of a dynamics, where the number of individuals in each functional subsystem evolves in time due to the transitions through functional subsystems.

The mathematical structure proposed in Sec. 3 is appropriate to take into account the aforementioned hallmarks, however, transferring the hints into specific models requires an important deal of work to be developed in a proper research program.

Finally, we wish to mention that while specific applications are (and will be) developed, some challenging analytic problems are brought to the attention of applied mathematicians. Two of them are worth to be mentioned among the various ones. The first one is the qualitative analysis of the initial-boundary value problem, where the results proposed in Ref. 7 for a crowd in unbounded domain might be further developed to take into account nonlocal interactions with walls and obstacles.

Similarly the derivation of macroscopic models from the underlying description delivered by kinetic models, proposed in Ref. 6 for unbounded domains, can be further developed in this specific case. Of course this challenging objective needs

further developments of the mathematical tools, proposed in Refs. 8 and 9, to take into account the various complex features of nonlocal and nonlinear interactions.

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